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Generalization of Exactness on Simple Ring

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***Abstract*: In this paper we will show that**

1. **Every simple ring is exact.**
2. **If R be any simple ring then each direct summand of $_R R$ and R_R is exact.**
3. **If R be any simple ring then any free left module over R is exact.**
4. **Let m, n $\in$ N then for any simple ring R the bimodule Hom $_R$ (R$^{(m)}$, R$^{(n)}$) is exact left M $_m$ (R) and right M $_n$ (R) bimodule.**
5. **Let n $\in$ N then for any simple ring R, End $_R$ (R$^{(n)}$) $\otimes$ End $_R$ (R$^{(m)}$)is exact.**
6. **For any simple ring R and any idempotent e in R, ReR≠0 is exact.**
7. **If R be any simple ring e be any idempotent in R then Hom $_R$ (Re $\underset{eRe}{\otimes}$ eR , R) and Hom $_R$ (eR $\underset{R}{\otimes}$ Re , R) are exact.**

Throughout this paper we will consider that all rings have unity and all modules are unitary.

***Keywords*: Simple Ring, Direct summand, free left module, Bimodule, Exact module, Idempotent.**

INTRODUCTION

In this paper we will study the exactness property of simple ring under the assumption that all rings have unity and all modules are unitary. Here we have rewritten the definition of simple ring such that it has a composition series of length one. Purpose of our work is to study the nature of simple ring .

Exactness:

Def: A bimodule that has a composition series whose composition factors are balanced is an exact bimodule, one sided module $_R M$ is exact in case bimodule $_R M_{End(_R M)}$ is exact, a ring R is called exact if regular bimodule $_R R_R$ is exact.

Simple ring:

Def: A ring R is called simple ring iff regular bimodule $_R R_R$ has a composition series

$$_R R_R = R \geq 0 = 0.$$

Theorem 1.1: Each simple ring is exact.

Proof: Let R be any simple ring then it does not have any ideals except 0 and R itself so regular bimodule $_R R_R$ does not have any bisubmodule except 0 and $_R R_R$ itself then it has composition series

$$_R R_R = R \geq 0 = 0,$$

here $R/0 \cong R$ is the only one composition factor and from Proposition 4.11[1] P 60

the regular bimodule $_R R_R$ is faithful and balanced therefore by the definition, ring R is an exact ring.

Corollary 1.2: If R is any simple ring then each direct summand of $_R R$ and R_R is exact.

Proof: Let R be any simple ring then from above Theorem, R is exact so by definition regular bimodule ${}_R R_R$ is exact then one sided modules ${}_R R$ and R_R are exact then from Lemma 2[4] each direct summand of ${}_R R$ and R_R are exact.

Corollary 1.3: If R be any simple ring then any free module over R is exact.

Proof: Given that R be any simple ring then from Theorem 1.1, R is exact and F be any free left module over R then,

$${}_R F \cong R^{(A)} \quad \text{for any } A \neq \emptyset$$

now from Lemma 2[4], F is exact.

Example1.4: Let R be any simple ring and let m, $n \in N$ then $Hom_R(R^{(m)}, R^{(n)})$ is an exact left M_m(R) and right M_n (R) bimodule.

Proof: Given that R be any simple ring and let m, $n \in N$ then $Hom_R(R^{(m)}, R^{(n)})$ is a left M_m(R) and right M_n(R) bimodule via the isomorphism

$$Hom_R(R^{(m)}, R^{(n)}) \cong M_{m\times n}(R).$$

Here $M_{m\times n}$(R) is a left M_m(R) and right M_n(R) bimodule and R is given simple ring so bimodule

$${}_{M_m(R)} M_{m\times n}(R)_{M_n(R)}$$

is a simple bimodule thus it has a composition series

$${}_{M_m(R)} M_{m\times n}(R)_{M_n(R)} = M_{m\times n}(R) \geq 0 = (0)$$

with composition foctor $M_{m\times n}$(R) which is balanced therefore $M_{m\times n}$(R) is exact and from Lemma 2 [4], $Hom_R(R^{(m)}, R^{(n)})$ is an exact left M_m(R) and right M_n(R) bimodule.

Example1.5: If R be any simple ring and let $n \in N$ then $End({}_R R^{(n)})$ is also exact.

Proof: If R be any simple ring then for any n>0

$$End({}_R R^{(n)}) = Hom_R(R^{(n)}, R^{(n)})$$

which is isomorphic to M_n(R) and since R is given simple so M_n(R) is also be a simple ring and from Theorem1.1, M_n(R) and consequently from Lemma2[4],

$End({}_R R^{(n)})$ is exact.

Example1.6: If m, $n \in N$ then for any simple ring R the tensor product

$End_R(R^{(m)}) \otimes End_R(R^{(n)})$ is exact.

Proof: Given that m, $n \in N$ then from Exercise6.22

$$End_R(R^{(m)}) \otimes End_R(R^{(n)}) \cong M_m(R) \otimes M_n(R)$$

now from above Example1.5, both $End_R(R^{(m)})$ and $End_R(R^{(n)})$ are exact and from Lemma 2[4], both M_m(R) and M_n(R) are exact and

$$End_R(R^{(m)}) \otimes End_R(R^{(n)}) = Hom_R(R^{(m)}, R^{(m)}) \otimes Hom_R(R^{(n)}, R^{(n)})$$

$$= \bigoplus_m Hom_R(R, R^{(m)}) \otimes \bigoplus_n Hom_R(R, R^{(n)})$$

$$\cong \bigoplus_m R^{(m)} \otimes \bigoplus_n R^{(n)}$$

$$\cong \bigoplus_n (\bigoplus_m R^{(m)} \otimes R^{(n)})$$

$$\cong \bigoplus_n \bigoplus_n (\bigoplus_m R^{(m)} \otimes R)$$

$$\cong \bigoplus_n \bigoplus_n (\bigoplus_m R^{(m)})$$

$$= \bigoplus_n \bigoplus_m (\bigoplus_n R^{(m)})$$

$$= \bigoplus_n \bigoplus_m R^{(n)(m)}$$

since

$$End_R\,(R^{(n)(m)}) \cong \bigoplus_n \bigoplus_m R^{(n)(m)}$$

and

$$End_R\,(R^{(n)(m)}) = End_R\,(R^{(nm)}) \cong M_{nm}(R)$$

here R is given simple so M_{nm} (R) is also simple and from Theorem1.1 it is exact ,

now from Lemma2[4], $End_R\,(R^{(m)}) \otimes End_R\,(R^{(n)})$ is exact.

Corollary1.6: Let R be any simple ring and e be any idempotent in R and if ReR≠0

then ReR is exact.

Proof: Given that R be any simple ring and e be any idempotent in R then eR_R and $_R Re$ are right and left ideals of R here ReR is the product of Re and eR and

we know that product of left and right ideal of any ring R is the two sided ideals of R, so ReR is the two sided ideals of R and since R is given simple so it has no two sided ideal of R except 0 and itself here ReR≠0 so ReR=R and R is given

simple then from Theorem1, R and consequently ReR is exact.

Theorem1.7: Let R be any simple ring and e be any idempotent in R then

$Hom_R\,(Re \otimes_{eRe} eR\,,\,R)$ and $Hom_R\,(eR \otimes_R Re\,,\,R)$ are exact.

Proof: Given that R be any simple ring and e is any idempotent in R, since here R is given simple ring then from Theorem1, R is exact so regular bimodule $_R R_R$ is exact so one sided module $_R R$ and R_R is exact, $_R Re$ and eR_R are left ideal and right ideal of R respectively then we consider a commutative diagram

$$\begin{array}{ccc} f: End_R\,(eR)=Hom_R\,(eR\,,\,eR) & \rightarrow & Hom_R\,(eR\,,\,Hom_R\,(Re\,,\,R)) \\ h\downarrow & & \downarrow Ø \\ Ø':\ eRe & \rightarrow & Hom_R\,(Re \otimes_{eRe} eR\,,\,R) \end{array}$$

from Proposition 4.6[1] P 58, f is an isomorphism and from Corollary 21.7[3], h is an isomorphism, here in above diagram Ø is an abelian group isomorphism (Z-isomorphism) then from 4.2 [1] P 56, f and h are Z-isomorphism therefore $\emptyset'$ is also be a Z-isomorphism and here eRe is the direct summand of ${}_R R_R$ and since R is exact so from Lemma 2[4], eRe and accordingly $Hom_R(Re \otimes_{eRe} eR, R)$ is exact,

now from Proposition21.6[3] and symmetricity of isomorphism we have,

$$\begin{array}{ccc} f' : End_R(Re) = Hom_R(Re, Re) & \rightarrow & Hom_R(Re, Hom_R(eR, R)) \\ h' \downarrow & & \downarrow \emptyset'' \\ \emptyset''' : eRe & \rightarrow & Hom_R(eR \otimes_R Re, R) \end{array}$$

here in above commutative diagram f', h' and $\emptyset''$ are Z-isomorphism so $\emptyset'''$ is also be Z-isomorphism, since eRe is the direct summand of ${}_R R_R$ which is exact by assumption therefore from Lemma 2[4], eRe and consequently abelian group $Hom_R(eR \otimes_R Re, R)$ is exact.

REFERENCES

[1] F. W. ANDERSON AND K. R. FULLER, Rings and Catagories of modules, New York Springer-Verlag Inc. 1973.

[2] LOUIS HALLE ROWEN, Ring Theory.

[3] T. Y. LAM, A first course in non commutative rings Springer-Verlag.

[4] WEIMIN XUE, Exact Modules and Serial Rings, Journal of Algebra 134, 209-221 [1990] .